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Order Statistics and Ginibre's Ensembles

B. Rider[1]

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The moduli of the eigenvalues at the edge of Ginibre's complex and quaternion Gaussian random matrix ensembles are shown to respond to a limit theorem identical in nature to that for independent identically distributed sequences. This is a companion work to ref. 15 in which the limit law for the (scaled) spectral radius of these ensembles was identified.

KEY WORDS: Random matrices; order statistics.

1. INTRODUCTION

Consider the ensembles of $N \times N$ random matrices in which all entries are independent and distributed as $\frac{1}{\sqrt{N}}$ times either a standard real, complex or quaternion Gaussian. These are the canonical non-Hermitian ensembles of Random Matrix Theory. They are widely referred to as Ginibre's ensembles, he being first to recognize their importance and, in the complex and quaternion cases, derive their explicit spectral density functions (ref. 10). The complex case is easiest to describe: the eigenvalues, labeled $z_1, z_2,..., z_N$ and lying in the complex plane $\mathscr{C}$, have distribution described by the joint density

$$P_N^C(z_1, z_2,..., z_N) = \frac{1}{Z_N^C} e^{-N \sum_{k=1}^N |z_k|^2} \prod_{1 \leqslant j < k \leqslant N} |z_j - z_k|^2. \tag{1.1}$$

The quaternion ensemble P_N^Q is absolutely continuous to P_N^C; its density has the additional factor $\prod_k |z_k - \bar{z}_k|^2 \prod_{j<k} |z_j - \bar{z}_k|^2$, causing the eigenvalues to be repulsed by the real axis in addition to the usual level-repulsion present

[1] Department of Mathematics, Duke University, Box 90320, Durham, North Carolina 27708; e-mail: rider@math.duke.edu

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in (1.1). The real case is more complicated still (we do not deal with it directly here, but see comments below).

While much of the recent explosion of activity in Random Matrix Theory has focused on Hermitian ensembles, there is a growing interest in the non-Hermitian and almost-Hermitian settings due to their relevance to various branches of theoretical physics (refs. 5 and 8 will direct you to the literature). In this note we complete the picture began in ref. 15 by determining limiting distributions for the edge eigenvalues, or actually their moduli. Define

$$\rho_\ell^{(N)} = \ell\text{th largest}\{|z_k| \text{ such that } z_k \text{ is an eigenvalue}\}$$

(in either the complex or quaternion ensemble), noting that $\rho_1^{(N)}$ is also the spectral radius. In accordance with the Circular Law[2] one expects that, for any fixed ℓ, these numbers will converge to 1 as $N \uparrow \infty$. A result of this type is found in ref. 9. We are interested in fluctuations and previously have proved:

Theorem (ref. 15). For $\rho_1^{(N)}$ the spectral radius of the $N \times N$ complex Gaussian ensemble we have

$$\lim_{N \uparrow \infty} P_N^C\left(\rho_1^{(N)} - 1 \leqslant \sqrt{\gamma_N/4N} + x/\sqrt{4N\gamma_N}\right) = e^{-e^{-x}} \tag{1.2}$$

in which $\gamma_N = \log N - (2 \log\log N + \log 2\pi)$. For the quaternion case (the P_N^Q measure) the result is the same up to a numerical factor: the right hand side of (1.2) is replaced by $e^{-\sqrt{2}\, e^{-x}}$.

The entertaining observation not pointed out in ref. 15 is that the right hand side of (1.2) is an *extremal distribution*: it is of the type exhibited by the maximum of suitably scaled independent variables. To explain, let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables, and bring in for each N their Order Statistics: $X_N^{(N)} \leqslant X_{N-1}^{(N)} \leqslant \cdots \leqslant X_1^{(N)}$ ($X_\ell^{(N)}$ is the ℓth largest of the first N observations). The classical fact (see ref. 12) is that if there exist sequences a_N and b_N such that $a_N X_1^{(N)} + b_N$ has a non-degenerate limiting distribution, then so does $a_N X_\ell^{(N)} + b_N$ (for any fixed ℓ) and it must be the case that

$$\lim_{N \uparrow \infty} P(a_N X_\ell^{(N)} + b_N \leqslant x) = H^{(\ell)}(x) := H(x) \sum_{k=0}^{\ell-1} \frac{1}{k!}\left[\log \frac{1}{H(x)}\right]^k \tag{1.3}$$

[2] This states that the empirical spectral distribution for all these models tends to the uniform measure on $|z| \leqslant 1$ as $N \uparrow \infty$, see for example ref. 1.

with a distribution function $H(x)=H^{(1)}(x)$ of one of the following three forms:

$$\begin{aligned} &\text{(i)} \quad e^{-e^{-x}} \text{ with support } (-\infty, \infty) \\ &\text{(ii)} \quad e^{-x^{-\gamma}} \text{ with support } [0, \infty) \text{ and } \gamma>0 \\ &\text{(iii)} \quad e^{-(-x)^{\gamma}} \text{ with support } (-\infty, 0] \text{ and } \gamma>0. \end{aligned} \tag{1.4}$$

An illustrative example is afforded by the case in which $X_1, X_2,...$ are uniformly distributed on $[0, 1]$. Then, without any scaling, $P(X_1^{(N)} \leqslant x) = x^N$ on $0 \leqslant x \leqslant 1$, $=0$ on $x<0$, and $=1$ on $x>1$, converges to a step function representing the degenerate distribution, or unit mass, at the point $x=1$. This type of thing is of no interest here. On the other hand, as $N \uparrow \infty$ the distribution function of the scaled variable $N(X_1^{(N)}-1)$ which equals $(1+x/N)^N$ on $-N \leqslant x \leqslant 0$ goes over into that of type (iii) with $\gamma=1$. More generally, ref. 12 contains conditions on the underlying distribution of the input sequence $\{X_\ell\}$ which allow (if they indeed exist) both the limiting type ((i), (ii), or (iii)) and the appropriate scaling (a_N and b_N) to be identified ahead of time.

Now, with the right hand side of (1.2) recognized as an H of type (i), our result may be anticipated:

Theorem 1.1. For any fixed ℓ we have that: with again $\gamma_N = \log N - (2 \log\log N + \log 2\pi)$,

$$\lim_{N \uparrow \infty} P_N^C(\rho_\ell^{(N)} - 1 \leqslant \sqrt{\gamma_N/4N} + x/\sqrt{4N\gamma_N}) = H^{(\ell)}(x) \tag{1.5}$$

where $H^{(\ell)}$ is as in (1.3) and H is of type (i) defined in (1.4). Once more the result extends to the quaternion ensemble with the same adjustment needed in (1.2)

Both (1.2) and Theorem 1.1 stem from the following product structure: taking the complex case and a radial test function $\phi(z)=\phi(|z|)$,

$$\int_{\mathscr{C}} \cdots \int_{\mathscr{C}} \phi(z_1) \cdots \phi(z_N)\, P_N^C(dz_1, dz_2,..., dz_N) = \prod_{\ell=1}^{N} E\left[\phi\left(\sqrt{\frac{\eta_1+\cdots+\eta_\ell}{N}}\right)\right] \tag{1.6}$$

in which $\{\eta_k\}$ form a sequence of independent exponentially distributed random variables of mean one. That the integral on the left factors is noted in Mehta[13] and utilized in refs. 5 and 6 (the latter also discuses certain edge statistics), but prior to ref. 15 we have not seen the right hand side cast in this more probabilistic light. The advantage of this viewpoint is that if ϕ concentrates in a neighborhood of $|z|=1$, the Central Limit Theorem immediately explains that it is only the last $O(\sqrt{N})$ terms which figure into (1.6) for N large. This observation motivates our derivation of (1.2) and (1.5). The physical picture is that the expected $O(\sqrt{N})$ eigenvalues outside of the unit disk are repulsed through their angular component, allowing their moduli to act more or less independently.

Remark 1.1. Our interest in the above line of questions derives in part from the well-known work of Tracy–Widom on the largest eigenvalue in $G(U/O/S)\,E$ (refs. 16 and 17). The structure here is far less intricate than for $G(U/O/S)\,E$, but this simplicity has its advantages, there being of yet no closed expression for the second, third, etc. largest eigenvalue distribution in those ensembles. However, see refs. 7, 18, and 19 for such results in the Laguerre ensemble, as well as the very recent ref. 3 which discusses this issue more generally.

Remark 1.2. That the result of Theorem 1.1 is not stated for the real Gaussian case lies in the complexity of that eigenvalue density (see ref. 4). We assume that this is just a technical roadblock and that the present result extends to a large class of non-Hermitian matrices with independent identically distributed entries. That is, the non-Hermitian edge statistics studied here should enjoy the type of universality known to hold in the Wigner case.[14]

Remark 1.3. Given how explicit the formulas are ((1.6) and below), another direction is to extend the type of result discussed here down into the bulk, computing fluctuations of $\rho^{(N)}_{\lceil \theta N \rceil}$ about its limit $\sqrt{\theta} \in (0,1)$. J. L. Lebowitz has kindly pointed out that such a computation has relevance to charge fluctuations in the two-dimensional one-component plasma (Jellium) model. In fact,[11] may be consulted for work on the corresponding Large Deviations of this problem.

2. PROOF OF THE THEOREM

Complex Ensemble. The probability that the ℓth eigenvalue (ordered by absolute value) lies within the disk of a given radius α is just the sum from $k=0$ to $\ell-1$ of the probabilities that there are exactly k

eigenvalues outside of that disk. The typical approach to the latter (gap probabilities) is to expand the right hand side of

$$\begin{aligned} &P_N^C(\text{exactly } k \text{ eigenvalues in } |z| > \alpha) \\ &\quad = \frac{1}{k!}\left(-\frac{d}{d\lambda}\right)^k E_N^C\left[\prod_{k=1,\ldots,N} (1-\lambda\chi_{[\alpha,\infty)}(|z_k|))\right]_{\lambda=1} \end{aligned} \tag{2.1}$$

employing the determinantal structure

$$P_N^C(z_1, z_2,\ldots, z_N) = \frac{1}{N!} e^{-N\sum_{k=1}^N |z_k|^2} \det[K_N(z_k, z_\ell)]_{1\leqslant k,\, \ell\leqslant N} \tag{2.2}$$

with kernel given by $K_N(z, w) = \frac{1}{\pi}\sum_{\ell=0}^{N-1} \frac{N^{\ell+1}}{\ell!} z^\ell \bar{w}^\ell$. Here, essential use is made of the fact that, as an operator, K_N projects onto the span of the first N polynomials orthogonal with respect to the weight $\mu_N(dz) = e^{-N|z|^2}\, d\Re(z)\, d\Im(z)$ on $\mathscr{C}$. On the other hand, the present non-Hermitian setup allows a simpler approach. Following Mehta,[13] appropriate row/column operations in the determinant (2.2) along with the symmetries of the E_N^C integrand (2.1) yield

$$\begin{aligned} &\frac{1}{N!}\int_{\mathscr{C}}\cdots\int_{\mathscr{C}} \prod_{k=1,\ldots,N} (1-\lambda\chi_{[\alpha,\infty)}(|z_k|)) \det[K_N(z_k, z_\ell)]_{1\leqslant k,\,\ell\leqslant N}\, \mu_N(dz_1)\cdots\mu_N(dz_N) \\ &\quad = \left(\pi^N \prod_{k=0,\ldots,N-1} k!\right)^{-1} \int_{\mathscr{C}}\cdots\int_{\mathscr{C}} \prod_{k=1,\ldots,N} (1-\lambda\chi_{[\alpha,\infty)}(|z_k|)) \\ &\qquad \times \det[\bar{z}_k^{k-1} z_k^{\ell-1}]_{1\leqslant k,\,\ell\leqslant N}\, \mu_N(dz_1)\cdots\mu_N(dz_N) \\ &\quad = \left(\pi^N \prod_{k=0,\ldots,N-1} k!\right)^{-1} \det\left[\int_{\mathscr{C}} (1-\lambda\chi_{[\alpha,\infty)}(|z|))\, \bar{z}^k z^\ell \mu_N(dz)\right]_{0\leqslant k,\,\ell\leqslant N-1} \\ &\quad = \prod_{k=0,\ldots,N-1} \left[1-\lambda\int_{\alpha^2}^{\infty} r^k e^{-Nr}\frac{dr}{k!N^{-(k+1)}}\right]. \end{aligned}$$

Now taking derivatives in λ as indicated in (2.1) and recognizing the probabilistic content of the integral in the last line of the previous display, we arrive at our basic formula:

$$\begin{aligned} &P_N^C(\text{exactly } k \text{ eigenvalues in } |z| \geqslant \alpha) \\ &\quad = P_N^C(\text{no eigenvalues in } |z| \geqslant \alpha) \\ &\qquad \times \sum_{1\leqslant \ell_1<\ell_2<\cdots<\ell_k\leqslant N} \left\{\prod_{m=1,\ldots,k} \frac{P\left(\frac{1}{N}\sum_{i=1}^{\ell_m} \eta_i \geqslant \alpha^2\right)}{P\left(\frac{1}{N}\sum_{i=1}^{\ell_m} \eta_i \leqslant \alpha^2\right)}\right\} \end{aligned} \tag{2.3}$$

where the $\{\eta_k\}$ form a sequence of independent exponential random variables of mean one.

It is interesting to mention the above in the context of the work of Borodin and Soshnikov[3] on Janossy densities for general determinantal ensembles. An instance of their result is that

$$
\begin{aligned}
&P_N^C(\text{exactly } k \text{ eigenvalues in } |z| \geqslant \alpha)\\
&\quad = P_N^C(\text{no eigenvalues in } |z| \geqslant \alpha)\\
&\quad \times \frac{1}{k!} \int_{|z_1| \geqslant \alpha} \cdots \int_{|z_k| \geqslant \alpha} \det[K_{N,\alpha}(z_i, z_j)]_{1 \leqslant i,j \leqslant k}\, \mu_N(dz_1) \cdots \mu_N(dz_k)
\end{aligned} \tag{2.4}
$$

in which the kernel $K_{N,\alpha}$ is defined similarly to K_N. It is again a projection onto the span of the first N orthogonal polynomials with respect to weight $\mu_N(dz)$, though now on the restricted domain $\{|z| < \alpha\}$. But note the monomials $\{z^k\}$ remain orthogonal on any radially symmetric domain with respect to either the present or any radially symmetric weight, and so $K_{N,\alpha}$ and K_N agree up to normalizers, making everything quite explicit. A bit of algebra will now take you from (2.4) to (2.3).

Returning to the proof, the right hand side of (2.3) is to be examined for $\alpha = \alpha_N(x) = 1 + \sqrt{\gamma_N/4N} + x/\sqrt{4N\gamma_N}$ (recall the statement) with fixed x and $N \uparrow \infty$. That the prefactor, the probability of all eigenvalues being of modulus less than $\alpha_N(x)$, settles down to the correct object is the old result (1.2) and so may be ignored. In fact, verifying (1.5) comes down to checking that

$$
\lim_{N \uparrow \infty} \sum_{1 \leqslant \ell_1 < \ell_2 < \cdots < \ell_k \leqslant N} \left\{ \prod_{m=1,\ldots,k} P\left(\frac{1}{N} \sum_{i=1}^{N-\ell_m} \eta_i \geqslant \alpha_N^2(x) \right) \right\} = \frac{1}{k!} e^{-kx} \tag{2.5}
$$

for whatever fixed integer k. Here the terms figuring in the denominator of (2.3) have been left out. That this may be done at the expense of multiplicative errors of order $1 + o(1)$ follows from: with x bounded and N large,

$$
\begin{aligned}
1 &\geqslant P\left(\frac{1}{N} \sum_{i=1}^{N-\ell} \eta_i \leqslant \alpha_N^2(x) \right) \geqslant P\left(\frac{1}{N} \sum_{i=1}^{N} \eta_i \leqslant 1 + \frac{1}{2}\sqrt{\log N/N} \right)\\
&\geqslant 1 - \exp\left[-N\left(\beta\left(1 + \frac{\sqrt{\log N}}{2\sqrt{N}}\right) + \log(1-\beta) \right) \right] \geqslant 1 - N^{-1/8}
\end{aligned}
$$

after an application of Chebychev's Inequality and choosing $\beta = \frac{1}{2}\sqrt{(\log N)/N}$ in the last step. On the other hand, pertaining to the remaining terms we have

$$P\left(\frac{1}{N}\sum_{i=1}^{N-\ell} \eta_i \geqslant \alpha_N^2(x)\right)$$
$$\leqslant \exp\left[-N\left(\beta + \left(1-\frac{\ell}{N}\right)\log(1-\beta)\right)\right] \leqslant \exp\left[-\frac{\ell^2}{2N}\right]$$

for $\beta = 1+(1+\ell/N)^{-1}$. Thus, any term in (2.5) containing an $\ell_m > 2\sqrt{kN\log N}$ decays like N^{-2k}. As there are $O(N^k)$ terms in total, it is allowed to reduce the sum in this manner.

Next, with the theorem being one of convergence in distribution the scaling function $\alpha_N(x)$ may be replaced with the more convenient approximation

$$\hat{\alpha}_N(x) = 1 + \frac{1}{2\sqrt{N}}\sqrt{\gamma_N + 2x} := 1 + \frac{f_N(x)}{2\sqrt{N}}.$$

It is also helpful to rescale within the probabilities making up the right hand side of (2.5) as in

$$P\left(\frac{1}{N}\sum_{i=1}^{N-\ell} \eta_i \geqslant \hat{\alpha}^2(x)\right) = P\left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N-\ell} (\eta_i - 1) \geqslant \phi_N(x) + \frac{\ell}{\sqrt{N}}\right) := p(\ell, N)$$

where we have made yet another definition: $\phi_N(x) := f_N(x) + \frac{1}{4\sqrt{N}} f_N^2(x)$. The result can now be explained by the Central Limit Theorem: with $\delta_N = 2\sqrt{k\log N}$,

$$\begin{aligned}
&\sum_{1\leqslant \ell_1 < \cdots < \ell_k \leqslant \sqrt{N}\,\delta_N} p(\ell_1, N)\cdots p(\ell_k, N)\\
&\quad\simeq \sum_{1\leqslant \ell_1 < \cdots < \ell_k \leqslant \sqrt{N}\,\delta_N} \prod_{m=1,\dots,k}\left[\int_{\phi_N(x)+\frac{\ell_m}{\sqrt{N}}}^{\infty} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\,du\right]\\
&\quad\simeq N^{k/2}\int_{\phi_N(x)}^{\infty}\int_{t_2}^{\infty}\cdots\int_{t_k}^{\infty}\prod_{m=1,\dots,k}\left[\int_{t_m}^{\infty}\frac{e^{-u^2/2}}{\sqrt{2\pi}}\,du\right]dt_1\cdots dt_k\\
&\quad= \frac{1}{k!}N^{k/2}\left(\int_{\phi_N(x)}^{\infty}\int_t^{\infty}\frac{e^{-u^2/2}}{\sqrt{2\pi}}\,du\,dt\right)^k = \frac{1}{k!}\left(\frac{\sqrt{N}\,e^{-f_N^2(x)/2}}{\sqrt{2\pi}\,f_N^2(x)}\right)^k(1+o(1)).
\end{aligned} \tag{2.6}$$

That the last line converges to $e^{-kx}/k!$ is readily checked. The proof is completed by demonstrating one has enough control of the relevant errors in order to make lines two and three rigorous.

The second line of (2.6), replacing each $p(\ell, N)$ by its Gaussian counterpart, relies on uniform corrections to the Central Limit Theorem courtesy the classical Edgeworth expansion. Letting $p_M(t)$ denote the density of the random variable $\frac{1}{\sqrt{M}}\sum_{i=1}^{M}(\eta_i - 1)$ at t, the statement (see ref. 2 Corollary 19.4) is that

$$\sup_{-\infty<t<\infty}\left|p_M(t) - \frac{e^{-t^2/2}}{\sqrt{2\pi}} - \frac{c_1 t^3 e^{-t^2/2}}{\sqrt{M}\sqrt{2\pi}}\right| = \mathrm{O}(M^{-1}) \tag{2.7}$$

with an explicitly computable constant c_1 independent of M. So, by first restricting the event in $p(\ell, N)$ such that the variable is less than $2\sqrt{\log N}$ as well as larger than $\phi_N(x)+\ell/\sqrt{N}$, employing (2.7), and then putting back the upper tail by a standard large deviation estimate, we have

$$p(\ell, N) = \int_{\phi_N(x)+\frac{\ell}{\sqrt{N}}}^{\infty} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\, dt + O\left(\frac{(\log N)^{3/2}}{N}\right) := g(\ell, N) + O\left(\frac{(\log N)^{3/2}}{N}\right).$$

Substituting this in the first expression in (2.6) it is easy to see that

$$\begin{aligned}
&\sum_{1\leqslant \ell_1<\cdots<\ell_k\leqslant\sqrt{N}\,\delta_N} p(\ell_1, N)\cdots p(\ell_k, N)\\
&\quad= \sum_{1\leqslant \ell_1<\cdots<\ell_k\leqslant\sqrt{N}\,\delta_N} g(\ell_1, N)\cdots g(\ell_k, N) + O\left(\frac{(\log N)^2}{\sqrt{N}}\right),
\end{aligned}$$

granted that the first expression on the right hand side is indeed of order one as $N\uparrow\infty$. That of course is the content of (2.6) once one checks that the error implicit in line three there is bounded (up to a constant depending on k) as in:

$$\begin{aligned}
&\left(N^{1/2}\sum_{1\leqslant\ell\leqslant\sqrt{N}\,\delta_N}\int_{\frac{\ell}{\sqrt{N}}}^{\frac{\ell+1}{\sqrt{N}}}\int_{\frac{\ell}{\sqrt{N}}}^{t} e^{-(\phi_N(x)+u)^2/2}\, du\, dt\right)^k \vee \left(N^{k/2}\int_{\delta_N}^{\infty} t^{k-1}e^{-t^2/2}\, dt\right)\\
&\quad= O\left((e^{-kf_N^2(x)/2}\delta_N^k)\vee(e^{-\delta_N^2/2}N^{k/2}\delta_N^{k-2})\right) = O\left(\left(\frac{\log N}{N}\right)^{k/2}\right),
\end{aligned}$$

the second equality valid for x restricted to a compact set. The proof is finished. ▮

Quaternion Ensemble. For Gaussian quaternion components an expression for P_N^Q averages of test functions of the form $\prod_{k=1,...,N} \phi(z_k)$ may be found in ref. 13 (p. 302):

$$\int_{\mathscr{C}} \cdots \int_{\mathscr{C}} \left[\prod_{k=1,\dots,N} \phi(z_k) \right] P_N^Q(dz_1,\dots, dz_N) = C_N \left[\det[\psi_{k\ell}(\phi)]_{0 \leqslant k,\, \ell \leqslant 2N-1} \right]^{\frac{1}{2}}.$$

Here the normalizer is $C_N = \prod_{k=1,...,N} (2N)^{2k}/\pi(2k-1)!$, and the matrix ψ is defined by

$$\psi_{k\ell}(\phi) = \int_{\mathscr{C}} e^{-2N\,|z|^2} (z-\bar{z})\, \phi(z)(z^k \bar{z}^\ell - z^\ell \bar{z}^k)\, d\Re(z)\, d\Im(z).$$

If $\phi(z) = \phi(|z|)$ then ψ reduces to a bi-diagonal: for $k-\ell = \mp 1$ we have $\psi_{k\ell} = \pm 2\pi \int_0^\infty \phi(r)\, e^{-2Nr^2} r^{k+\ell+2}\, dr$ and $\psi_{k\ell} = 0$ otherwise. That determinant is easily seen to be a perfect square and one finds that:

$$\begin{aligned} E_N^Q \left[\prod_{k=1,\dots,N} \phi(|z_k|) \right] &= \prod_{k=1,\dots,N} \left[\frac{(2N)^{2k}}{(2k-1)!} \int_0^\infty \phi(\sqrt{r})\, e^{-2Nr} r^{2k-1}\, dr \right] \\ &= \prod_{k=1,\dots,N} E\left[\phi\left(\sqrt{\frac{\eta_1 + \cdots + \eta_{2k}}{2N}} \right) \right], \end{aligned}$$

in which the $\{\eta_k\}$ are once again independent exponential random variables of mean one. That the proof in the complex case may then be repeated verbatim save for the adjustment of a few constants is quite plain. ∎

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